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Designing Covariation Tasks to Support Students' Reasoning about Quantities involved in Rate of Change

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Johnson ThemeB: Designing Covariation Tasks to Support Students' Reasoning about Quantities involved in Rate of Change

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This paper articulates theoretical and practical considerations for designing a sequence of covariation tasks to support students' reasoning about quantities involved in rate of change. Adapting the well-known bottle problem during two iterations of implementation and analysis, this researcher-developed task sequence incorporates dynamically linked geometric and graphical representations of covarying quantities and prompts fostering students' coordination of quantities that are changing together. Taking into account students' perspective of quantities involved, this task sequence is designed to support students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.

Task design, Reasoning, Quantity, Covariation, Rate of Change

Researchers using mathematical tasks involving dynamic representations of covarying quantities have supported middle and high school students' consideration of relationships between quantities involved in rate of change (e.g., Johnson, 2012b; Monk & Nemirovsky, 1994; Saldanha & Thompson, 1998; Stroup, 2002). In this paper I articulate theoretical and practical considerations for designing a sequence of covariation tasks to support middle school students' reasoning about quantities involved in rate of change. The design of this task sequence accounted for students' perspective of quantities involved to support their progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity.

Background

Drawing on Sierpinska's (2004) characterization, by *mathematical task* I mean a purposefully designed problem intended for a particular audience. By problem, I mean a situated problem (Gravemeijer, 1994) involving a particular context. I take an interpretive stance on context, drawing on Van Oers' articulation of context: "What counts as context depends on how a situation is interpreted in terms of activity to be carried out" (1998, p. 481). Acknowledging that individuals for whom a task is intended can interpret the task in myriad ways, I assume that an individual's perspective on the nature of the problem to be solved can influence an individual's reasoning about mathematics he or she perceives to be involved in the task.

I consider mathematical reasoning to be an individual's purposeful mental activity situated within a particular context. The purposeful activity includes making sense of how a mathematical situation holds together (Simon, 1996), making relationships between objects involved in a situation (Thompson, 1996), and engaging in operation that involves carrying out actions both mentally and physically (Piaget, 1970). When characterizing reasoning as quantitative, I consider a quantity to be an individual's conception of the measurability of an attribute of an object (Thompson, 1994). Because individuals do not need to determine actual measurements to reason quantitatively, quantitative mental operations are nonnumerical (Thompson, 1994). By articulating that the object of the reasoning is quantities involved in rate of change, I do not assume that individuals will reason about rate of change as single entity. Focusing on covariation (Carlson et al., 2002), I attend to how individuals make sense of and make relationships between quantities that are changing together.

Adapting the well-known bottle problem to design covariation tasks

I designed covaration tasks by adapting the well-known bottle problem developed by the Nottingham University's Shell Centre (Swan & the Shell Centre Team, 1999). Given the context of a bottle filling with liquid being dispensed into the bottle at a constant rate and a picture of a bottle, the bottle problem requires students to sketch a graph of the changing height of the liquid as a function of the changing volume. Researchers have used the bottle problem to investigate the reasoning of undergraduate and graduate mathematics students (Carlson et al., 2002) and prospective elementary (Carlson, Larsen, & Lesh, 2003) and secondary (Heid, Lunt, Portnoy, & Zembat, 2006) mathematics teachers. My adaptations to the bottle problem have had two iterations of implementation and analysis. The task sequence reported in this paper is from the second iteration.

In the first iteration I developed a covariation task by adapting the bottle problem in two ways: (1) Providing students with a graph and asking students to sketch a bottle that the graph could represent, and (2) Using a graph that represented the changing volume of the liquid as a function of the changing height. Prompts included in this covaration task were: (1) How is the volume of the liquid in the bottle changing as the height of the liquid in the bottle increases? (2) Sketch a bottle that the graph could represent. Intending to implement the task with high school students who had not yet taken a calculus course, I provided students with a graph because previous research (Carlson et al., 2002; Heid et al., 2006) found that even students with extensive mathematics background have difficulty creating graphs. I chose to represent volume as a function of height in part because preservice elementary teachers working on the bottle problem operated with the independent variable, volume, as if it were time (Carlson, Larsen, & Lesh, 2003). I hypothesized that representing volume as a function of height might reduce the likelihood of students treating the independent variable as if it were time.

In the second iteration I adapted Thompson, Byerly, and Hatfield's (in press) version of the bottle problem that dynamically links a pictorial representation of a filling bottle with a graph relating the volume of liquid in the bottle to the height of liquid in the bottle. Key to Thompson et al.'s adaptation was the use of a dynamic environment linking pictorial and graphical representations. I hypothesized that such an environment could foster students' reasoning about covarying quantities by explicitly linking a change in one representation with a change in another representation. Hence, I adapted Thompson et al.'s bottle problem (implemented with beginning calculus students) for use with 7th grade pre-algebra students. Anticipating that 7th grade students might have limited conceptions of volume, I altered the context of the task from a filling bottle to a two-dimensional shape being filled with area. In making this change, the two dimensional pictorial representation would represent a two dimensional rather than a three dimensional quantity.

The adaptation for 7th grade pre-algebra students resulted in a sequence of four tasks. Accompanying each task was a dynamic sketch I developed using Geometer's

Sketchpad software (Jackiw, 2009). The filling rectangle sketch (see Fig. 1) linked a rectangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the rectangle by animating or dragging point H. Students could drag point F to vary the width of the rectangle, then predict and create corresponding graphs representing the amount of shaded area as a function of its height. The filling triangle sketch (see Fig. 2) linked a right triangle with a graph that related the amount of shaded area to the height of the shaded area. Students could vary the height of the triangle by animating or dragging point D, then predict and create a corresponding graph representing the amount of shaded area as a function of its height. By affording students' manipulation of dynamically linked representations, the dynamic sketches seem to foster students' consideration of relationships between covarying quantities.

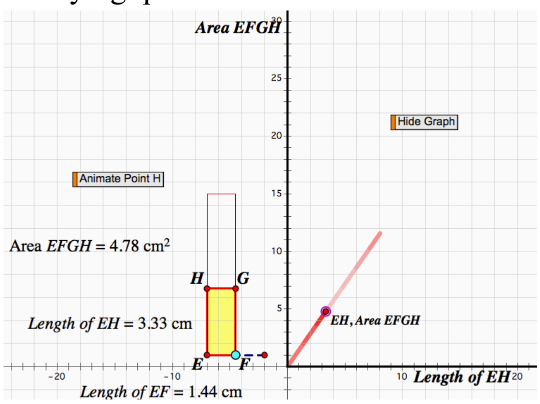

Figure 1. Filling Rectangle Sketch

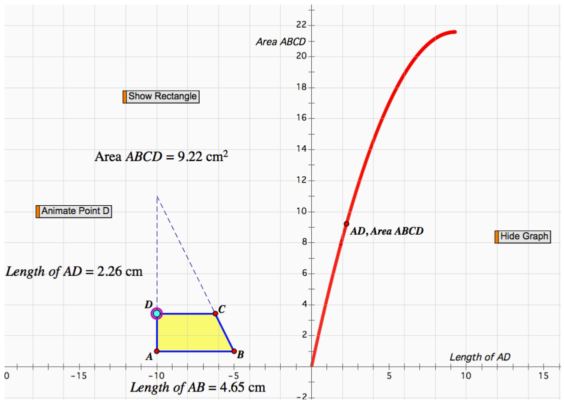

Figure 2. Filling Triangle Sketch

Task sequence

The task sequence is designed to support students' progression in using nonnumerical quantitative reasoning to coordinate covarying quantities. The description of the task sequence includes: (1) Statement describing the context for the task sequence (2) Identification of dynamic sketch used with each task (filling rectangle or triangle), (3) Quantitative reasoning (QR) objective for each task *(italics)*, and (4) Prompts fostering students' coordination of quantities that are changing together. The context for the task sequence describes the situation on which the tasks are based. A QR objective is distinct from a learning objective because it indicates purposeful activity intended to support a way of reasoning rather than an intended mathematical understanding. These QR objectives indicate purposeful ways of making sense of and making relationships between covarying quantities. Prompts refer to questions and directives designed to foster students' making of relationships between quantities and predictions about characteristics of linked representations.

Context for task sequence: Imagine that a shape (rectangle/triangle) is being filled with area that is increasing at a constant rate.

1. Task #1: Filling Rectangle Sketch. *Create and use non well-ordered tables of values indicating measurements of related quantities to predict characteristics of graphs relating those quantities.*
    a. Press *Animate Point* to run the animation of the filling rectangle. What changes and what stays the same?
    b. Given a non-well ordered table of heights for a rectangle with a given base, determine the different amounts of area. For example:

Imagine the side length of EF was **4 cm**. Complete the table:

Length of Side EH	1 cm	3 cm	5 cm	7 cm	10 cm
Area of Rectangle EFGH					

c. Imagine you created a graph relating the side length of EH and the filling area of EFGH. Would the graph be linear? Explain why or why not.
d. Repeat b&c for bases of different lengths. How would the graphs be similar/different?

2. Task #2: Filling Rectangle Sketch. *Use an amount of change in one quantity to predict an amount of change in a related quantity.*
   a. Given a non-well ordered table of heights for a rectangle with a given base (e.g., the table in 1b), determine the different amounts of area.
   b. Determine at least two different ways to complete this statement: When the height increases by ______, the area increases by ______. How many ways can this statement be completed?
   c. When given the two different heights for the same base, determine the amount of increase in area. For example,

   Imagine the side length of EF was **3 cm**. Complete the table:

Length of side EH	14.5 cm	16.5 cm
Amounts of increase in area of rectangle EFGH		

3. Task #3: Filling Rectangle Sketch. *(1) Use a graph relating two quantities to predict a measure for a third, related quantity (not represented explicitly by the graph). (2) Use a relationship between two quantities to predict characteristics of a graph relating one of those quantities to a third quantity.*
   a. Given graphs representing the amount of filling area as a function of the length of side EH, predict the length of the base of the rectangle.
   b. Given a relationship between changing area and changing height (e.g., *As the length of EH increases by 2 cm, the area of rectangle EFGH increases by 3* cm^2*)*, predict what a graph relating area and the side length of EH would be like.
   c. What would a graph be like for a rectangle with (1) a very short base, (2) a very long base? Why?
4. Task #4: Filling Triangle Sketch. *Use a dynamic geometric representation of covarying quantities to predict characteristics of a graph relating the quantities.*
   a. Press *Animate Point* to run the animation of the filling triangle. What changes and what stays the same?
   b. How does area change as the height increases?
   c. Imagine you created a graph relating the side length of AD and the filling area ABCD. What would the graph be like?
   d. Press *Animate Point* to sketch the graph. Was it what you expected? Why do you think it looks that way?

Design principles

Three principles guided my design of the task sequence. Central to each of these principles was my consideration of how students might perceive the nature of the quantities involved and how students' perspectives might influence their consideration of relationship between those quantities.

(1) Anticipate students' perspectives on relationships between changing quantities

Drawing on results of analysis of students' work on the first adaptation of the bottle problem, I anticipated two distinct perspectives students might have on relationships between change in the covarying quantities involved in the task sequence: (a) Quantities can change simultaneously with change in one quantity being independent of change in a related quantity (Johnson, 2012a) and (b) Change in one quantity depends on change in a related quantity (Johnson, 2012b).

(a) By relating covarying quantities as if each quantity were changing independently of the other quantity with respect to time, students can make comparisons between amounts of change in each quantity (Johnson, 2012a). For a filling rectangle, if 7th grade students could determine amounts of area for given amounts of height (The height being the length of EH – see Fig. 1), they could make comparisons between amounts of change in area and amounts of change in height. By sequencing filling rectangle tasks that afforded calculation of amounts of area prior to a filling triangle task that problematized calculation of amounts of area, I intended to support students' gradual move away from making numerical calculations.

(b) Reasoning about change in one quantity as being dependent on change in a related quantity can support students' attention to variation in the intensity of a change (Johnson, 2012b). If 7th grade students were able to envision the area of the filling triangle as varying in relationship to the height (The height being the length of AD – see Fig. 2), they could make claims about the area of the triangle increasing more slowly as the height increased. By requiring students to predict how area would change as height increased when manipulating dynamically linked geometric and graphical representations, I intended to support students' reasoning about variation in area as being dependent on variation in height.

(2) Incorporate key aspects to support students' attention to covarying quantities

The task sequence incorporated three key aspects: (a) Tables that were not well ordered, (b) Questions supporting students' attention to covarying quantities, and (c) Questions supporting students' making of predictions about change.

(a) In the filling rectangle tasks I incorporated tables that were not well ordered. By indicating that a table is not well ordered, I mean that the independent variable contained in the table (in the case of the filling rectangle tasks, the length of EH) does not increase by a uniform amount. When working only with well ordered tables, secondary students did not necessarily attend to change in both independent and dependent variables (Lobato, Ellis, & Munoz, 2003) and tended to pay attention to numerical patterns (Ellis, 2007). I anticipated that including tables that were not well ordered could promote students' attention to and making of relationships between covarying quantities.

(b) Both the filling rectangle and the filling triangle tasks incorporated prompts supporting students' attention to and making of relationships between the changing quantities of area and height. The prompt "What changes and what stays the same?" provided an entry point into the task and fostered students' attention to different quantities involved in the task. Subsequent prompts supported students' making relationships between quantities that were changing together. For example, in task 2b, completing the statement "When the height increases by ______, the area increases by ______" in multiple ways supports students' consideration of multiple relationships between amounts of increase in height and area for a rectangle with a

given base. Once students could begin to focus on relationships between quantities, it seems reasonable that students could then draw on those relationships to engage in related activity.

(c) Each task supported students' activity of making predictions about (1) characteristics of a linked representation (tasks 1, 3, 4) or (2) a related amount of change (task 2). Making predictions can foster students' use of nonnumerical reasoning by supporting students' envisioning of running through calculations without actually making the calculations to make relationships between the changing area and the changing height. For example, task 2c required students to predict amounts of change in area given two different heights without actually determining amounts of area. If students determined amounts of area, subtracted those amounts, then arrived at amounts of change, it could indicate that students were not yet relating changes in height with changes in area. By requiring students to predict characteristics of graphs representing rectangles with differently sized bases, task 3c supported students' use of relationships between area and height. Such predictions were intended to support students' attention to relationships rather than to results of calculations.

(3) Sequence tasks to support students' progression from numerically based reasoning to nonnumerically based reasoning

I sequenced the filling rectangle tasks prior to the filling triangle task based on two research hypotheses related to students' perspectives on quantities involved: (a) Students can use numerical calculations (with or without actually engaging in calculations) to make comparisons between amounts of change in covarying quantities, and (b) By focusing on covariation, students can attend to situations involving constant rate of change in ways that can support their attention to variation in the intensity of change in situations involving varying rates of change.

(a) While numerical calculations can be an entry point into the filling rectangle tasks, the design of each filling rectangle task is not intended to support students generalizing from numerical calculations. In contrast, intent is to support students' engagement in the nonnumerical operation of linking a changing area with a changing height. Students working from numerical calculations could begin to imagine running through calculations (without actually completing the calculations) to relate the changing area to the changing height.

(b) The filling rectangle tasks incorporate constant rates of change, and the filling triangle task incorporates a rate of change that increases at a decreasing rate. Although it may seem obvious to position tasks involving constant rate of change task prior to a task involving varying rate of change, research has questioned whether situations involving constant rate of change are sufficiently complex to engender students' reasoning related to varying rate of change (Stroup, 2002). Filling rectangle tasks incorporating constant rate of change supported students' focus on the covarying quantities of volume and height. I anticipated that such tasks would contain sufficient complexity to support students' consideration of situations involving varying rate of change because they could foster students' coordination of covarying quantities.

Task implementation and analysis

During May 2012, I implemented the filling rectangle tasks with 4 sections of 7th grade students at an urban middle school in a large Midwestern U.S. city. The district has identified the school as high performing based on students' academic performance, with approximately 45% of students identified as English Language

Learners and over 90% of students receiving free or reduced lunch. I implemented tasks 1-3 during three consecutive days of whole class instruction. Following the lessons, I conducted 40-minute task-based interviews with 7 pairs of students, selecting at least 1 pair of students from each of the 4 sections. I purposefully chose student pairs based on the students' participation in classroom instruction and on evidence of reasoning about quantities involved in rate of change. During the task-based interviews I followed up with task 3 and presented task 4. For this paper I report results of analysis of students' work on task 4.

Analysis of students' work during task-based interviews revealed two main findings: (a) Students may depend on numerical calculations to make claims about how quantities are changing together, and (b) Students may create graphs relating covarying quantities (not including time) as if one quantity were elapsing time.

(a) Students who depended on numerical calculations had difficulty making predictions about how the area of the triangle would change as the height increased. The responses of two students, Navarro and Myra (who participated in different interview pairs), provide insight into the kind of difficulty students might have. When Navarro and Myra were presented with the filling triangle task, both of them attempted to determine amounts of area. Even after prompting to not worry about making calculations, Navarro's persistence in trying to calculate amounts of area made it seem as if he depended on calculating amounts of areas to make such predictions. Unlike Navarro, after my prompt to not worry about how to calculate the area, Myra smiled and exclaimed "Oh, I get you now!" When I asked her to explain, she said "the area is getting bigger, but how much it increases is getting smaller." By no longer attempting to determine amounts of area, Myra was able to describe variation in how the area was increasing. Future iterations of implementation and analysis could provide further explanation as to how students' nonnumerical reasoning develops when making relationships between quantities.

(b) Although the filling triangle sketch related area with the length of AD, when predicting features of a graph some students seemed to operate as if AD represented time. I designed the animation (See Fig. 2) so that the graph would begin to be sketched from a point when the triangle was partially filled. When predicting the shape of a graph relating area and side length for the filling triangle, Tomas sketched a graph in the air, indicating the graph would curve and then begin to increase again (which is not consistent with the graph shown in Fig. 2). When asked to explain why, Tomas said that area would fill more slowly and then start filling more quickly again. Even though Tomas attended to both area and side length when using the animation, he seemed to sketch the graph as if it were relating increasing area with elapsing time. Future tasks dynamically linking multiple graphs relating changing quantities to a single geometric representation might support students' consideration of the independent variable as something other than time.

Implications for task design in research investigating students' reasoning

Designing a sequence of tasks to support students' reasoning involved theoretical considerations including how students might make sense of and coordinate covarying quantities and practical considerations including how students might manipulate quantities represented with dynamic geometry sketches. The task sequence supports students' progression in using nonnumerical quantitative reasoning to make predictions and create representations indicating how one quantity might change in relationship to another changing quantity. Researchers could draw on and expand design principles underlying this task sequence to develop other task sequences focusing on quantity and covariation. Future research involving implementation of this and other task sequences could support development and expansion of frameworks articulating progressions in quantitative and covariational reasoning.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33(5), 352–378.

Carlson, M. P., Larsen, S., & Lesh, R. A. (2003). Integrating a models and modeling perspective with existing research and practice. In R. A. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 465-478). Mahwah, NJ: Lawrence Erlbaum Associates.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439-478.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443-471.

Heid, M. K., Lunt, J., Portnoy, N., & Zembat, I. O. (2006). Ways in which prospective secondary mathematics teachers deal with mathematical complexity. In S. Alatorre, J. L. Cortina, M. Sáiz & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 2-9). Mérida, Mexico.

Jackiw, N. (2009). *The Geometer's Sketchpad (Version 5.0) [Computer Software]*. Emerville, CA: Key Curriculum Technologies.

Johnson, H. L. (2012a). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 39-53). Laramie, WY: University of Wyoming College of Education.

Johnson, H. L. (2012b). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. Journal of Mathematical Behavior, 31(3), 313-330.

Lobato, J., Ellis, A. B., & Munoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning, 5*(1), 1-36.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I* (pp. 139-168).

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson, K. R. Dawkins, M. Blanton, W. N. Coloumbe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the Psychology of Mathematics Education North American Chapter* (Vol. 1, pp. 298-303). Raleigh, NC: North Carolina State University.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197-209.

Stroup, W. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning, 7*, 167-215.

Swan, M. & the Shell Centre Team. (1999). The language of functions and graphs. Nottingham, U.K.: Shell Centre Publications.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 267-284). Mahwah, NJ: Lawrence Erlbaum Associates.

Thompson, P. W., Byerley, C., & Hatfield, N. (in press). A conceptual approach to calculus made possible by technology. *Computers in the Schools*.

van Oers, B. (1998). From context to contextualizing. *Learning and Instruction, 8*(6), 473-488.