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DEVIN'S CONSTRUCTION OF A MULTIPLICATIVE DOUBLE COUNTING SCHEME: FOCUS ON DUAL ANTICIPATION OF START AND STOP

Nicola M. Hodkowski
University of Colrado Denver, nicola.hodkowski@ucdenver.edu

Ron Tzur
University of Colrado Denver, ron.tzur@ucdenver.edu

Rachael Risley
risleyrachael@gmail.com

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Rachael Risley | Nicola M. Hodkowski | Ron Tzur

University of Colorado, Denver

risleyrachael@gmail.com | nicola.hodkowski@ucdenver.edu | ron.tzur@ucdenver.edu

In this case study with Devin (pseudonym), which was part of a larger, constructivist teaching experiment with students identified as having learning difficulties in mathematics, we examine how a fourth grader constructed a dual anticipation involved in monitoring when to start and when to stop the simultaneous count of composite units (numbers larger than 1) in multiplicative tasks. We postulate that such a dual anticipation underlies the first, Multiplicative Double Counting (mDC) scheme (Tzur et al,. 2013) that marks children's conceptual progress from additive to multiplicative reasoning. Data from two teaching episodes with Devin focus on his anticipation of the start/stop features of his double counting activity. We discuss theoretical implications of these findings in terms of similarity between the dual anticipation in additive and multiplicative reasoning, and practical implications in terms of task design and sequencing.

Keywords: *Multiplicative Reasoning, Learning Difficulties*

Introduction

Researchers have developed models for thinking about and promoting children's learning to reason multiplicatively (Park & Nunes, 2001; Sophian & Madrid, 2003; Steffe, 1992, 1994; Steffe & Cobb, 1994; Tzur et al., 2013). By and large, these models are rooted in studies of normally achieving peers' (NAPs) construction of particular multiplicative schemes. Currently, less is understood about how students identified as having learning difficulties (SLDs) construct understandings of multiplicative reasoning (Evans, 2007). To address this lacuna, our study addressed the problem: How may SLDs progress *within* the identified scheme of multiplicative Double Counting—the first in their transition from additive to multiplicative reasoning?

In particular, we examine a possible intermediate stage in SLDs' construction of the mDC scheme, which may be a corollary of the intermediate stage distinction of anticipating starts and stops in the additive operation of counting on (Tzur & Lambert, 2011). Specifically, they found that students who learn to count need to develop an anticipation of both a starting and stopping point for the double count needed. For example, a child who is adding 7+4 would anticipate both a starting point of the number after 7 and the need to double count each item added as being simultaneously a constituent of the sequence of added items (1-2-3-4) and of the total (8-9-10-11). That is, in counting on, the child's counting of the added items serves as a purposeful aid in knowing when to stop her activity. In this paper, we focus on a similar anticipation of knowing where to start and where to stop counting of composite units which can be inferred in SLDs' work.

Conceptual Framework

Our conceptual framework for this study consists of general and content-specific constructs. The general constructs are rooted in a constructivist perspective on knowing and learning (von Glasersfeld, 1995). We view one's transition from not knowing to knowing a new mathematical idea as a mental activity that an observer must infer from observable actions and language of learners. Specifically, we drew on the construct of scheme as a three-part mental structure (von

Glasersfeld, 1995). The first part of a scheme is the recognition of a certain situation. Here, a learner uses assimilation to "recognize" the situation based on previously recorded, like experiences. This "recognition template" sets the person's goal. In turn, the goal triggers the second part of the scheme, a mental activity used to accomplish the goal that may or may not be coupled with observable actions. The third part of the scheme is the result a learner expects to follow the activity based on previous, similar activities.

Building on this three-part notion of scheme and on Piaget's (2001) core notion of reflective abstraction, Simon, Tzur, Heinz and Kinzel (2004) articulated cognitive change as the mental process of reflection on activity-effect relationship (Ref*AER). This mental process is postulated to take place as a learner compares between the expected result and actual effect of an activity, and through comparison across instances of similar activity-effect dyads. Learning is marked by a change in the relationship between an activity and newly noticed effects, which turn into a new anticipation that can be linked to different situations.

The content-specific constructs we used are based on Tzur et al.'s (2013) developmental framework of six schemes used by children to reason multiplicatively. They distinguished those schemes based on units and operations a child uses. Specifically, scheme for multiplicative reasoning are thought of in terms of units coordinating activities a child is inferred to be using (Norton, Boyce, Ulrich, & Phillips, 2015; Steffe, 1994; Steffe & Cobb, 1994). In this paper we focused on the first scheme, termed multiplicative Double Counting (mDC).

A child who has constructed the mDC scheme anticipates the effect of coordinating at least two levels of composite units (CU). For example, such a child anticipates that 12 could be composed by coordinating an operation in which, say, 4 units of 1 (e.g., 4 cubes per tower) are distributed over the items of another composite unit (e.g., 3 towers) (Tzur et al., 2013). Consider, for example, a child who is presented with a multiplicative situation such as, "There are six packs of gum, each with 4 pieces of gum; how many pieces of gum are there in all six packs together?" A student who can bring forth and use the mDC scheme might sequentially hold up six fingers while simultaneously distribute four pieces of gum across to each (e.g., the first is 4, the second is 8, etc.), and arrive at the answer of 24 pieces of gum.

Research has suggested several possible reasons that SLDs may not make adequate progress in mathematics, including reliance on counting based strategies longer than NAPs (Geary & Hoard, 2003), memory issues (Raghubar, Barnes, & Hecht, 2010), and SLDs' lack of the developmental requisite of composite unit (Tzur, Xin, Si, Kenny and Guebert, 2010) to construct multiplicative reasoning. However, it should be noted that different researchers define mathematical disabilities in various ways. In our study we drew on a definition that includes both students who may have "qualified" for specialized support given to those identified as having mathematics learning disability as well as students who are significantly behind in mathematics achievement (Mazzocco, 2007). Throughout the rest of the paper, we will use the term students with learning difficulties (SLD) to refer to students who are significantly behind in mathematics—whether or not they were identified by their school system as such.

Methodology

The case study on which we report in this paper was part of a larger constructivist teaching experiment (Cobb & Steffe, 1983) conducted with four 4th graders in a western US school as part of the first author's doctoral dissertation. The students were sampled for the instructional intervention based on underachievement measured on state mathematics assessments along with their classroom teacher recommendations. The two first authors conducted the video recorded teaching episodes with the group of students or with individual students. We focus in this paper on teaching episodes conducted with one student, Devin (pseudonym), twice a week (30-45 minutes each), from October through December of 2014.

To teach and study Devin's construction of the mDC scheme, in each episode the researcher-teachers engaged him in playing a version of the game, *Please Go and Bring for Me* (PGBM), which Tzur et al. (2013) described in detail. The PGBM game was designed to promote children's reflection on the units used in multiplicative reasoning, by asking them to build towers from a given number of single cubes. The game is played in pairs (with either a peer or the teacher). The players take turns as either a "Sender" or a "Bringer." The Sender poses the task by asking the bringer to build and bring back several, same-size towers, one tower at a time. For example, the Sender may plan to ask the bringer to bring 3 towers with 5 cubes; she would first ask the bringer to construct and bring one tower of 5, then another tower of 5 cubes, etc. Once the bringer brought all towers to the Sender's satisfaction, the Sender asks the Bringer four questions (in our work – those were written on a poster to promote students' use of full sentences and explicit mention of units): (a) How many towers did you bring (emphasizes number of composite units)? (b) How many cubes are in each tower (emphasizes unit rate – number of 1s in each composite unit)? (c) How many cubes are in all the towers? (d) How did you figure this [total of 1s] out? Similarly, the poster included 'answer-starters' that enabled the bringer to express her answers as full sentences (e.g., "I brought __ towers"). Initially, the teacher constrained the game so Devin could only use particular numbers of cubes per tower (e.g., 2 or 5) and of towers in all (e.g., up to 6 towers). The teacher also asked Devin to use different numbers for each kind of unit (e.g., disallow bringing 5 towers of 5 cubes each).

Our line-by-line retrospective analysis of video records, transcripts, and researcher field notes taken during each episode focused on the third and fifth teaching episodes with Devin. The focus of the analysis was on transitions Devin might make within the multiplicative Double Counting (mDC) scheme, while 'zooming in' on the interplay between his ways of operating and the numbers chosen in each task. The two first authors conducted ongoing analysis following each episode. The entire team of authors then conducted the line-by-line analysis of the three segments presented in the next section.

Results

In this section we present and analyze three excerpts that demonstrate advances in Devin's construction of goal-directed activities of coordinated counting to solve multiplicative tasks in which we used what for him were harder numbers (e.g., 5 towers of 6 cubes each and 7 towers of 6 cubes each, symbolized as 5T6 and 7T6, respectively). We selected these data because, prior to Devin's work on these tasks, to solve PGBM tasks with "easier" numbers he could independently call up an anticipation suitable for figuring out the total of 1s in a compilation of CUs via a coordinated count of the dual accrual of 1s and CUs. However, he was yet to construct and independently use an anticipation of the need to monitor his simultaneous count of composite units to know where to stop the count when "harder" numbers were given.

Excerpt 1 presents how Nina (pseudonym, the teacher/researcher and second author of this paper) began the work with Devin in the typical way PGBM is played, particularly when towers are hidden, namely, asking him to repeat (and firmly establish) the number of towers and the number of cubes per tower. This common practice when playing the game can help eschew a claim that the child's difficulties result from short-term memory issues. Excerpt 1 shows Devin's facility in the anticipation required for the double counts in the accumulation of up to 5 towers, and his inability to anticipate where to stop in the accumulation of the unit rates.

Excerpt 1, keeping track of where to stop when counting the unit rate (student: Devin; task 5T6; date: October 15, 2014).

11:21 Nina: How many towers are there, Devin?

11:28 Devin: There are s…. There are … how many towers? Five towers.

11:30 Nina: How many cubes are in each tower?

11:32 Devin: There are six cubes in each tower.

11:34 Nina: [Covers the towers with a piece of paper after Devin has confirmed the number of towers and the number of cubes per tower.] How many cubes are there in all?

11:45 Devin: [Under his breath.] Six; twelve; [He presses the thumb and index finger of his right hand on the table to indicate the first two towers, then shifts to counting six 1s with his left hand starting with his thumb.] 13, 14, 15, 16, 17, 18; [He presses the third finger of his right hand on the table and continues to incorrectly count five 1s using the fingers on his left hand.] 19, 20, 21, 22, 23; [He then presses the fourth finger of his right hand and continued to count five more 1s.] 24, 25, 26, 27, 28; [He then presses the fourth finger of his right hand and continues to count six more] 29, 30, 31, 32, 33, 34. [He raises his hand indicating that he has finished and is ready to give his answer.]

Devin's response to the question about how many cubes are there in all indicated an anticipation of the need to coordinate the counts of 1s and CUs. Specifically, he independently initiated a start of the counting from the second multiple of 6, indicating his coordination of the compilation of CUs and the unit rate. That is, with the first two easy (for him) numbers in the sequence of multiples of 6, he seemed to distinguish the number of 1s in each CU from the number of composite units for which he had been accounting so far. As he tracked the accumulation of cubes past these two easy numbers (6, 12), however, he shifted to counting only five (instead of six) items per composite unit while keeping track of six towers (instead of just five). That is, whereas with "easy" numbers the role of each unit in regulating Devin's activity was properly anticipated, with "hard" numbers his effort to focus on accrual of both types of unit took over these roles. In this sense, Devin provided an example of a student for whom a claim that the more difficult (5+n) numbers impact construction of mDC can be demonstrated.

Excerpt 2 provides data from an instructional prompt from Nina designed to capitalize on Devin's work (and errors). In this excerpt, Nina oriented Devin's reflection to the way in which he accurately used his left hand to keep track of six cubes.

Excerpt 2, teacher/researcher supporting tracking the unit rate (student: Devin; task 5T6; date: October 15, 2014).

12:42 Devin: There are 34 cubes altogether.

12:57 Nina: Now I want you to double check.

13:00 Devin: [Uncovers the cubes and starts counting at 12 cubes for two towers (confirming his mDC even when the cubes were available to him). Moves the first two towers and continues counting the rest of the towers by 1s, arriving at 30. Smiles and looks at Nina.] I got it wrong.

13:10 Nina: [Orienting Devin on his own tracking methods.] So, Devin; what I saw you doing, which I thought was so good, is this. [Nina holds her left hand over the table and begins to fold each finger down.] You had this hand and were going like this [Holds her left hand over the table and begins to fold each finger down.] Can you tell me what you were doing?

13:24 Devin: [Places both hands on the table and moves his right hand.] That [hand] was towers [moves his left hand] and this [hand] was [for] my cubes.

13:26 Nina: [Models Devin's error.] And what I saw you doing was something like this. [Folds over each of 5 fingers on her right hand]
13:34 Devin: Oh, it was 5 [Indicates possible recognition of his error of counting 5 cubes instead of 6 cubes.]
13:37 Nina: So you were doing… almost like… [Gestures at Devin to show his count.] Can you show me?
13:42 Devin: Um, oh so 5 towers [presses his right hand on the table] and cubes [presses his left hand on the table] so 6+6 is 12.
13:56 Nina: So how many towers is that?
13:57 Devin: [Demonstrates his ability to track towers.] Two.
13:58 Nina: Ok.
13:59 Devin: So my third tower is [presses each finger of his left hand on the table and counts his thumb twice for an accurate 6 count.] 13, 14, 15, 16, 17, 18.
14:06 Nina: Ok … [Does not indicate whether the response is correct or incorrect, but encourages him to continue.]
14:08 Devin: My fourth tower is: 19, 20, 21, 22, 23, 24.
14:13 Nina: Ok …
14:15 Devin: My fifth tower is: 25, 26, 27, 28, 29, 30.
14:20 Nina: [Compares Devin's answer to another student's response they both heard previously.] So you got 30, too.

Excerpt 2 shows that, with Nina's prompting, Devin appeared to recognize the inconsistency in his previous counting of the unit rate in this task (13:34). Furthermore, he was able to re-orient his attention to systematically monitoring the subsequent stops of each of the unit rate counts at 6. We assert that when Nina oriented Devin's reflection on his own tracking methods, she fostered his monitoring of his own goal-directed activity. Consequently, Devin could begin monitoring a stop at 6 for each tower as he monitored the accrual of the total number of cubes. A crucial point in the shift in Devin's counting activity is that, for each tower, he *first* stated its number in the sequence of accruing CUs and only then interjected the 1s that constituted that CU. This is a subtle but important difference from first counting the 1s and then raising a finger for the CU, in that the count of 1s after stating the CU's ordinal number may indicate distribution of those items into each of the CUs (as shown in the study by Clark and Kamii, 1996).

Excerpt 3 provides data from Devin's work on a task during an episode that took place three teaching episodes after the one presented in Excerpts 1 and 2. In Excerpt 3, Devin still needed prompting to regulate his stoppage of the count of the unit rate when solving mDC tasks with harder numbers. This provides additional evidence of the difficulty of developing the double anticipation (start, stop) required for an mDC when faced with larger numbers. Despite of Devin's acceptance of Nina's suggestion, and his subsequent ability to monitor the stops in the unit rate for 5T6 in Excerpt 2, Devin's anticipation of where to stop counting the unit rate remained inconsistent. This inconsistency suggests that, in the previous episodes, Devin's construction of the coordinated count was at the participatory (prompt-dependent) stage.

In Excerpt 3, Devin first determined the total number of cubes in 7 towers of 6 cubes in each tower (7T6). In this case, we purposely increased the difficulty of the numbers because both—the compilation of CU and the unit rates—exceeded the number of fingers (five) on each of his hands. Devin readily anticipated where to start both counts, but again was unable to regulate where to stop the unit rate count past the second tower.

Excerpt 3, anticipating the stop when tracking the unit rate (student: Devin; task 7T6; date: October 23, 2014).

15:24 Devin: [Raises the thumb and index finger of his right hand] So 6+6 is 12; [then] 13, 14, 15, 16, 17, 18 [Raises the middle finger of his right hand and continues counting.] 19, 20, 21, 22, 23, 24 [Raises the ring finger of his right hand and continues counting albeit with 5 counts this time.] 25, 26, 27, 28, 29, [Raises the pinkie finger of his right hand and continues counting 5 counts again.] 30, 31, 32, 33, 34 [He folds all fingers on his right hand and raises his thumb again. At this point, the thumbs of his right hand (towers) and left hand (cubes) are raised.] He counts six counts.] 34, 35, 36 37, 38, 39…. I got 39 but I got 40 before.

Excerpt 3 indicates that, in this task, Devin could independently initiate the coordinated count while beginning from the second (known) multiple of 6, just as he had in Excerpt 2. This independent initiation indicated his anticipated coordination of the compilation of CUs and the unit rate. That is, with the first two "easy" (for him) numbers in the sequence of multiples of 6, he seemed to clearly distinguish and monitor the number of 1s in each CU and the number of CUs for which he had been accounting so far. As he tracked the accumulation of cubes for the fifth and sixth multiples of 6, however, he shifted to counting only five (instead of six) items per CU. At the seventh multiple, he recounted 34 (the last number he said when incorrectly counting the 6th multiple), and then accurately counted 6 counts of 1s. That is, whereas Devin's prompt-dependent anticipation of when to start/stop each count with "hard" numbers lagged behind his independent, correct use of this anticipation with "easy" numbers.

Discussion

In this paper we examined a difficulty SLDs may face when learning to anticipate both where to start and where to stop in each of the coordinated counts required in mDC. The difficulty to anticipate this dual monitoring illustrates a possible intermediate stage in the development of mDC, particularly by SLDs. mDC requires a coordination of more than one count and each of the coordinated counts requires a dual anticipation of where to start and where to stop. The analysis of Devin's work in this study amounts to suggesting that mDC requires the learner's (and teacher's) attention to both an anticipation of where to shift each count of a CU (unit rate, first anticipation) and of where to stop a count for the compilation of CUs (second anticipation). It is this dual anticipation that makes operating on harder numbers, for example, when CUs and/or 1s exceed the number of fingers on one hand, a challenging feat to overcome. We suggest that this difficult conceptual shift may be rooted in the need to count figural items standing for each type of unit through a 1-to-1 correspondence (e.g., finger and the unit for which it stands).

In addition, Devin was able to recall the number of towers and the number of cubes per tower, but not yet to anticipate the stops when operating on those units. Thus, his case indicated that besides memory issues, other, conceptually born factors may underlie challenges SLDs' face when constructing the mDC for any number. That is, Devin could clearly and independently remember and accurately continue a coordinated count from the 2nd multiple of six. With prompting, he could also distinguish the number of 1s in each CU and the number of CUs for which he had been accounting so far (2T6). However, Devin's action in the subsequent coordinated counts highlighted the ongoing difficulty of anticipating stops within the operation of a double count in the development of mDC. This finding seems consistent with Tzur and Lambert's (2011) identification of two intermediate sub-schemes in children's progress from counting all to counting on in additive situations. Specifically, it suggests a possible extension to include anticipation of the start and stop of two simultaneous counts in multiplicative situations.

References

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. Journal for Research in Mathematics Education, 27(1), 41-51.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14*(2), 83-94.

Evans, D. (2007). Developing mathematical proficiency in the Australian context: Implications for students with learning difficulties. *Journal of Learning Disabilities, 40*(5), 420-426.

Geary, D.C., & Hoard, M.K. (2003). Learning disabilities in mathematics: Deficits in memory and cognition. In J.M. Royer (Ed.), Mathematical Cognition (pp. 93-116). Greenwich, CT: Information Age.

Mazzocco, M. M. M. (2007). Definining and differentiating mathematical learning disabilities and difficulties. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*. Baltimore: Paul H. Brookes.

Norton, A., Boyce, S., Ulrich, C., & Phillips, N. (2015). Students' units coordination activity: A cross-sectional analysis. *The Journal of Mathematical Behavior, 39*, 51-66.

Park, J.-H., & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development, 16*, 763-773.

Piaget, J. (1986). Equilibration processes in the psychobiological development of the child. In H. E. Gruber & J. J. Voneche (Eds.), *The essential Piaget* (pp. 832-841). New York, NY: Basic Books.

Piaget, J. (2001). *Studies in reflecting abstraction.* Sussex, England: Psychology Press.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual and cognitive approaches. *Learning and Individual Differences, 20*(2), 110-122.

Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305-329.

Sophian, K., & Madrid, S. (2003). Young children's reasoning about many- to-one correspondences. *Child Development, 74*(5), 1418-1432.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-40). Albany, NY: State University of New York Press.

Steffe, L. P., & Cobb, P. (1994). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics, 16*(1 and 2), 45-61.

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. E., Xin, Y. P., Si, L., . . . Jin, X. (2013). Distinguishing schemes and tasks in children's development of mulitplicative reasoning. *PNA, 7*(3), 85-101.

Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as ZPD correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education, 42*(5), 418-450.

Tzur, R., Xin, Y. P., Si, L., Kenney, R., & Guebert, A. (2010). *Students with learning disability in math are left behind in multiplicative reasoning? Number as abstract composite unit is a likely 'culprit'*. Paper presented at the American Educational Research Association.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning* (Vol. 6). London: Falmer.